M303 Further Pure Mathematics

Exam Friday 12 June 2020

Unofficial Solutions - these have NOT been checked by the module team. They will likely contain mistakes and omissions but may be helpful for revision purposes.

(Student version)

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Q 1.

(a)

(i) By the Euclidean Algorithm, we obtain:

$$\begin{aligned} 132 &= 1 \times 77 + 55 \\ 77 &= 1 \times 55 + 22 \\ 55 &= 2 \times 22 + 11 \\ 22 &= 2 \times 11 + 0. \end{aligned}$$

Hence hcf(132,77) is equal to the last non-zero remainder, namely 11.

(ii) By backwards substitution

$$\begin{aligned} 11 &= 55 - 2 \times 22 \\ &= 55 - 2 \times (77 - 55) \\ &= 3 \times 55 - 2 \times 77 \\ &= 3 \times (132 - 77) - 2 \times 77 \\ &= 3 \times 132 - 5 \times 77 \end{aligned}$$

Thus $\text{hcf}(132, 77) = 132s + 77t$ where $s = 3, t = -5$.

(b) $x \equiv 1 \pmod 3$, $2x \equiv 1 \pmod 7$, $x \equiv 6 \pmod{10}$

By Thm 1.2, HB p19 we have

$$\begin{aligned} 2x \equiv 1 \pmod 7 &\Leftrightarrow 2x \equiv 8 \pmod 7 \\ &\Leftrightarrow x \equiv 4 \pmod 7 \end{aligned}$$

and

$$x + 1 \equiv 7 \pmod{10} \Leftrightarrow x \equiv 6 \pmod{10}$$

So by inspection,

$$\begin{aligned} x &\equiv 1 \pmod 3, \text{ gives } 1, 4, 7, 10, 13, 16, 19, 22, \ldots \\ x &\equiv 4 \pmod 7, \text{ gives } 4, 11, 18, 25, 32, 39, 46, \ldots \\ x &\equiv 6 \pmod{10}, \text{ gives } 6, 16, 26, 36, 46, \ldots \end{aligned}$$

As $46 \equiv 1 \pmod 3$, the least positive integer that satisfies all three of the given congruences is 46.

(c) As 3,7,10 are coprime then lcm(3,7,10)=210 so the congruence class is

$$46 \pmod{210}.$$

Q 2.

$G = \mathbb{Z}_4 \times \mathbb{Z}_{11}$

(a)

(i) $|G| = 4 \times 11 = 44$.

'Write down' so no reason needed, but Prop 2.10 HB p 32 applies.

(ii) The order of (0,3) in G is 11.
(Since 0 has order 1 in $\mathbb{Z}_4$, 3 has order 11 in $\mathbb{Z}_{11}$ and lcm(1, 11) = 11.)

Again 'Write down' so "11" gets the mark.

(iii) (2,3) has order 22 in G.
(In fact any element of the form $(2, k)$, with $1 \leq k \leq 10$ will do.)

(iv) Yes, G is cyclic.
(Since 4 and 11 are coprime, by Thm 2.11, HB, p32.)

(b) $D_{22} = \langle r, s \,|\, r^{22} = s^2 = e, sr = r^{21}s \rangle$

(i) We know that $sr^{-i} = r^i s$ (HB, p34). This, together with the fact that $r^{-11} = r^{11}$:

$$r^{11}sr^{11} = r^{11}sr^{-11} = r^{11}\left(r^{11}s\right) = r^{22}s = s.$$

So, to put $r^{11}sr^{11}$ in the form $r^i s^j$, we take $i = 0, j = 1$.

(ii) We have

$$\begin{aligned}\left(r^{11}s\right)^2 &= r^{11}sr^{11}s \\ &= \left(r^{11}sr^{11}\right)s \\ &= ss \text{ by part (i)} \\ &= e\end{aligned}$$

It follows that $r^{11}s$ has order 2.

(iii) Yes, $\langle r^{11} \rangle$ is normal in D_{22}.
We have $\langle r^{11} \rangle = \{e, r^{11}\}$. Further
$sr^{11}s^{-1} = sr^{11}s = ssr^{-11} = r^{11}$
and
$rr^{11}r^{-1} = r^{11}$
So the result follows from Prop 3.7(a), HB p27

Q 3.

(a) $28 = 2^2 \times 7$. The divisors of 28 are 1, 2, 4, 7, 14 and 28.

(i) By Defn 2.2 (The τ function, M303 HB, p17), since 28 has 6 distinct factors including 1 and 28 we have $\tau(28) = 6$.

(ii) By Prop 1.7 (The prime decomposition formula for σ, M303 HB, p42) we have

$$\sigma(28) = (1 + 2 + 2^2)(1 + 7) = 7 \times 8 = 56.$$

Alternatively, we could simply sum the numbers in the list of divisors above.

(iii) By Alternative formula for $\phi(n)$ (M303 HB, p44) we have

$$\phi(28) = 2^{(2-1)}7^{(1-1)}(2 - 1)(7 - 1) = 2 \times 1 \times 1 \times 6 = 12.$$

(b) Using Thm 2.2, Prop 3.2, Thm 4.2 and Prop 4.7 (M303 HB, p46) we have the following:

(i) $(2/199) = 1$ (by Prop 3.2 since $199 \equiv 7 \pmod 8$).

(ii)

$$\begin{aligned}
(23/199) &= -(199/23) \text{ (by Thm 4.2 since } 23 \equiv 199 \equiv 3 \pmod 4) \\
&= -(15/23) \text{ (by Thm 2.2(a) since } 199 \equiv 15 \pmod{23}) \\
&= -(3/23)(5/23) \text{ (by Thm 2.2(c) since } 15 = 3 \times 5) \\
&= -1 \times 1 \times (5/23) \text{ (by Prop 4.7 since } 23 \equiv (-1) \pmod{12}) \\
&= -1 \times 1 \times (23/5) \text{ (by Thm 4.2 since } 5 \equiv 1 \pmod 4) \\
&= -1 \times 1 \times (3/5) \text{ (by Thm 2.2(a) since } 23 \equiv 3 \pmod 5) \\
&= -1 \times 1 \times -1 = 1 \text{ (by Prop 4.7 since } 5 \equiv 5 \pmod{12}).
\end{aligned}$$

Hence $(23/199) = 1$.

(c) We have $f(x) = x^3 - 4x^2 + 8x + 2$. If a cubic is reducible there there will be a linear factor. So we need to see if $f(x)$ has a root in $\mathbb{Q}$ in order to see if it is reducible over $\mathbb{Q}$.

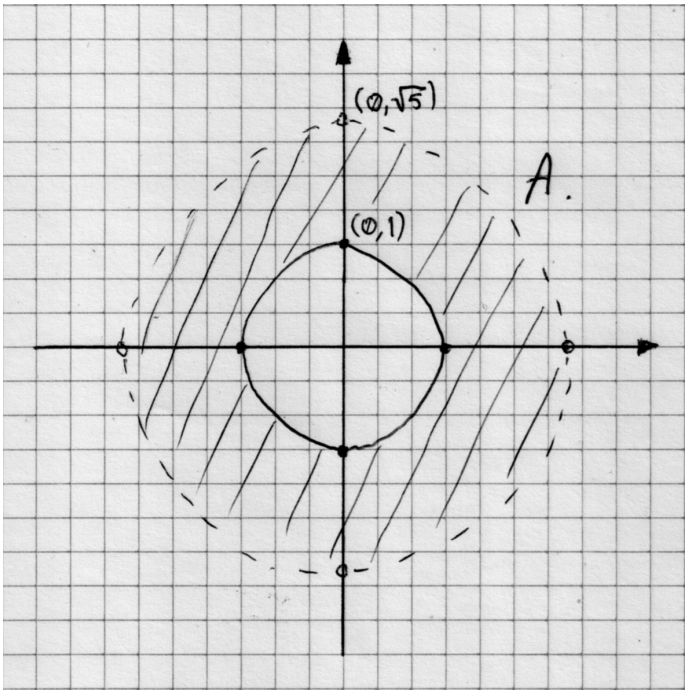
By the Rational Root Test (Thm 4.7, M303 HB, p52), any rational root $\frac{p}{q} \in \mathbb{Q}$ of $f(x)$ with hcf(p, q)=1 must satisfy $p \mid a_0 = 2$ and $q \mid a_3 = 1$. The possible values for p are therefore ± 1 and ± 2 and the possible values for q are ± 1. Since the denominator of any potential root is ± 1 the only possible roots of $f(x)$ in $\mathbb{Q}$ are ± 1 and ± 2. Testing these solutions we have:

$$\begin{aligned}
f(-1) &= (-1)^3 - 4(-1)^2 + 8(-1) + 2 = (-11) \neq 0 \\
f(1) &= 1^3 - 4 \times 1^2 + 8 \times 1 + 2 = 7 \neq 0 \\
f(-2) &= (-2)^3 - 4(-2)^2 + 8(-2) + 2 = (-38) \neq 0 \\
f(2) &= 2^3 - 4 \times 2^2 + 8 \times 2 + 2 = 10 \neq 0
\end{aligned}$$

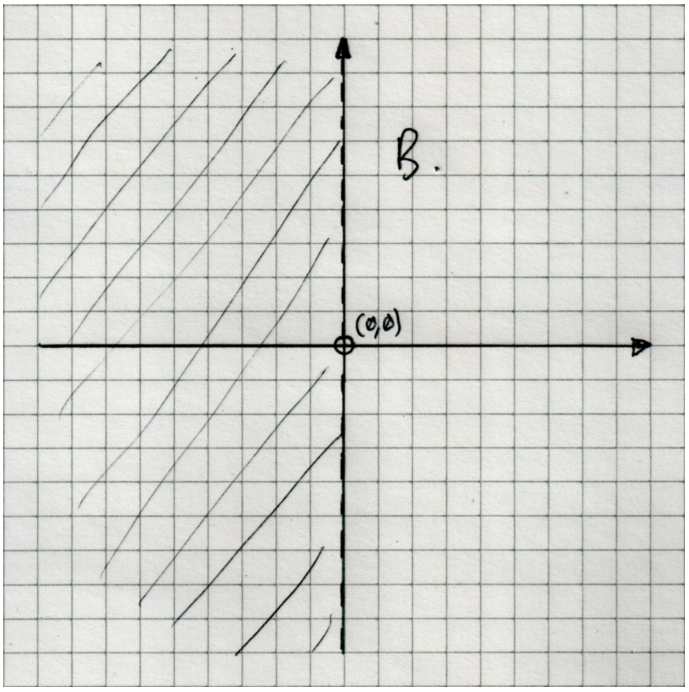
Since none of the values $\pm 1, \pm 2$ is a rational root of $f(x)$, it follows that $f(x)$ is irreducible over $\mathbb{Q}$ by the Rational Root Test, as required.

Q 4.

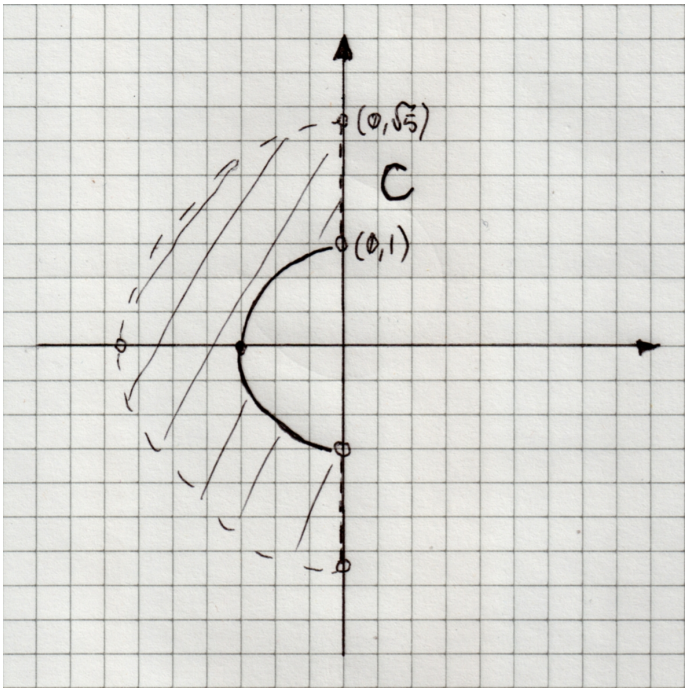
(a) Sketches of the sets A, B and C



A is the set of points in $\mathbb{R}^2$ that lie between the circles of radius 1 and radius $\sqrt{5}$ centered on the point $(0,0)$, including the points on the unit circle, but not the points on the circle of radius $\sqrt{5}$.



B is the set of points in $\mathbb{R}^2$ in the left half-plane excluding all points with $a = 0$.



C is the set of points in $\mathbb{R}^2$ that lie in the left half-plane between the circles of radius 1 and radius $\sqrt{5}$ centered on the point $(0,0)$, including the points on the unit circle with $a < 0$ and exluding all points on the circle of radius $\sqrt{5}$ and any points with $a = 0$.

(b) By Definition 3.6 (Closure point, HB, p84) the point $(0,1) \in \mathbb{R}^2$ is a $d^{(2)}$-closure point of C in $\mathbb{R}^2$ if for each $r > 0$, $B_d((-1,0), r) \cap C \neq \emptyset$. We can see from the diagram above that any open ball around $(0,1)$ will meet the set C, for instance, $(0 - r, 1) \in C$.
Hence $(0,1)$ is a closure point of C in $\mathbb{R}^2$ for the Euclidean metric.

(c) By Definition 4.5 (Interior point, HB, p85), the point $(-1,0) \in \mathbb{R}^2$ is a $d^{(2)}$-interior point of C in $\mathbb{R}^2$ if there is an $r > 0$ so that $B_{d^{(2)}}((-1,0), r) \subseteq C$. Again, from the diagram, we can see that any open ball around the point $(-1,0)$ will extend outside of C, for instance $(-1 + r, 0) \notin C$. Hence, $(-1,0)$ is not an interior point of C in $\mathbb{R}^2$.

(d)

$$\begin{aligned}
\mathrm{Cl}_{d^{(2)}}(C) &= \{(a,b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 5 \text{ and } a \leq 0\} \\
\mathrm{Int}_{d^{(2)}}(C) &= \{(a,b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 5 \text{ and } a < 0\} \\
\mathrm{Bd}_{d^{(2)}}(C) &= \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1 \text{ and } a \leq 0\} \\
&\quad \cup \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 5 \text{ and } a \leq 0\} \\
&\quad \cup \{(0,b) \in \mathbb{R}^2 : 1 \leq |b| \leq \sqrt{5}\}.
\end{aligned}$$

Q 5.

$\alpha = \sqrt{11} - \sqrt{3}$ and we have that $[\mathbb{Q}(\sqrt{11}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$, $\sqrt{11} \notin \mathbb{Q}(\sqrt{3})$ and $\sqrt{3} \notin \mathbb{Q}(\sqrt{11})$.

(a)

$$\begin{aligned}\alpha^3 &= (\sqrt{11} - \sqrt{3})^3 \\ &= (\sqrt{11} - \sqrt{3})(11 - 2\sqrt{11}\sqrt{3} + 3) \\ &= (\sqrt{11} - \sqrt{3})(14 - 2\sqrt{11}\sqrt{3}) \\ &= 14\sqrt{11} + 6\sqrt{11} - 14\sqrt{3} - 22\sqrt{3} \\ &= 20\sqrt{11} - 36\sqrt{3}.\end{aligned}$$

Hence $\alpha^3 = a\sqrt{11} + b\sqrt{3}$ with $a = 20$ and $b = -36$, both in $\mathbb{Q}$.

(b) If we consider $\alpha^3 - 36\alpha$ then by part (a) we have

$$\begin{aligned}\alpha^3 - 36\alpha &= 20\sqrt{11} - 36\sqrt{3} - 36(\sqrt{11} - \sqrt{3}) \\ \alpha^3 - 36\alpha &= -16\sqrt{11} \\ \sqrt{11} &= \frac{36\alpha - \alpha^3}{16}.\end{aligned}$$

Since $\sqrt{11}$ can be written solely in terms of α and elements of $\mathbb{Q}$, (namely, $36, -1, 1/16$), we have that $\sqrt{11} \in \mathbb{Q}(\alpha)$. Similarly, if we now consider $\sqrt{11} - \alpha$ we have

$$\begin{aligned}\sqrt{11} - \alpha &= \sqrt{11} - (\sqrt{11} - \sqrt{3}) \\ \sqrt{3} &= \frac{36\alpha - \alpha^3}{16} - \alpha.\end{aligned}$$

Here we have $\sqrt{3}$ solely in terms of α and $\mathbb{Q}$ and so $\sqrt{3} \in \mathbb{Q}(\alpha)$.
Hence, since both can be expressed in terms of elements of $\mathbb{Q}$ and α, we have $\sqrt{11}, \sqrt{3} \in \mathbb{Q}(\alpha)$ as required.

(c) By part (b) we have that $\sqrt{11}, \sqrt{3} \in \mathbb{Q}$, so $\mathbb{Q}(\sqrt{11}, \sqrt{3}) \subseteq \mathbb{Q}(\alpha)$. We also have $\alpha = \sqrt{11} - \sqrt{3}$. Since α can be expressed in terms of elements of $\mathbb{Q}$, (in this case, ± 1), $\sqrt{11}$ and $\sqrt{3}$, this means that $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{11}, \sqrt{3})$. Since we have $\mathbb{Q}(\sqrt{11}, \sqrt{3}) \subseteq \mathbb{Q}(\alpha)$ and $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{11}, \sqrt{3})$, $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{11}, \sqrt{3})$, as required.

(d) We have that $\sqrt{3} \notin \mathbb{Q}(\sqrt{11})$, which means that the minimal polynomial for $\sqrt{3}$ over $\mathbb{Q}(\sqrt{11})$ is $x^2 - 3$. Hence $[\mathbb{Q}(\sqrt{11}, \sqrt{3}) : \mathbb{Q}(\sqrt{11})] = 2$.

By the Theorem 2.29 (KLM Theorem for field extensions, HB, p105) we now have

$$\begin{aligned}[\mathbb{Q}(\sqrt{11}, \sqrt{3}) : \mathbb{Q}] &= [\mathbb{Q}(\sqrt{11}, \sqrt{3}) : \mathbb{Q}(\sqrt{11})] \times [\mathbb{Q}(\sqrt{11}) : \mathbb{Q}] \\ &= 2 \times 2 = 4.\end{aligned}$$

By part (c), $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{11}, \sqrt{3})$, so the degree of $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is 4.

(e) By Theorem 2.19 (M303 Handbook, p104) and part (d), the minimal polynomial for α over $\mathbb{Q}$ is monic and of degree 4. We have

$$\begin{aligned}
\alpha^2 &= (\sqrt{11} - \sqrt{3})^2 \\
\alpha^2 &= 14 - 2\sqrt{11}\sqrt{3} \\
\alpha^2 - 14 &= -2\sqrt{11}\sqrt{3} \\
(\alpha^2 - 14)^2 &= (-2\sqrt{11}\sqrt{3})^2 \\
\alpha^4 - 28\alpha^2 + 196 &= 4 \times 11 \times 3 \\
\alpha^4 - 28\alpha^2 + 196 &= 132 \\
\alpha^4 - 28\alpha^2 + 64 &= 0.
\end{aligned}$$

Since this polynomial is of degree 4 and is clearly monic, the minimal polynomial for α over $\mathbb{Q}$ is $x^4 - 28x^2 + 64$.

Q 6.

We have a metric d on $X = \mathbb{R}^2 - \{(a,b) \in \mathbb{R}^2 : b = 0\}$ given by

$$d(x,y) = 3\,|x_1 - y_1| + |x_2 - y_2|$$

(a) $d\big((-1,1),(1,1)\big) = 3|-1-1| + |1-1| = 3 \times 2 + 0 = 6$

(b) A d-disconnection of X is $\{U,V\} = \Big\{ \{(a,b) \in \mathbb{R}^2 : b < 0\}, \{(a,b) \in \mathbb{R}^2 : b > 0\} \Big\}$.

(c)

(i) By the definition of the metric we have, for any $(a,b) \in X$ and any $n \in \mathbb{N}$,

$$\begin{aligned} d\Big((0,1/n),(a,b)\Big) &= 3\,|0-a| + |1/n - b| \\ &= 3\,|a| + |1/n - b| \\ &\geq |1/n - b| \end{aligned}$$

Similarly, $d\Big((0,1/n),(a,b)\Big) \geq 3\,|a|$ and hence we have

$$d\Big((0,1/n),(a,b)\Big) \geq 3\,|a| \geq \max\{3\,|a|, |1/n - b|\} \tag{6.1}$$

Our aim is to show that there is no possible limit in X for the given sequence $\Big((0,1/n)\Big)$.
By Defn 3.2 (HB, p73), for $(a,b) \in X$, if $(0,1/n)$ converges to (a,b) then $\Big(d\big((1/n,0),(a,b)\big)\Big)$ is a null sequence.
First we consider the case $a \neq 0$, then, by (6.1),

$$d\Big((0,1/n),(a,b)\Big) \geq \max\{3\,|a|, |1/n - b|\} \geq 3\,|a|$$

from which it follows that $\Big(d\big((1/n,0),(a,b)\big)\Big)$ is not a null sequence.
Second we consider the case $a = 0$, noting that, as $(a,b) \in X$, $b \neq 0$. Further

$$d\Big((0,1/n),(a,b)\Big) \geq \max\{3 \times 0, |1/n - b|\} \geq |1/n - b|\,.$$

and since $|1/n - b| \to |b| > 0$ as $n \to \infty$ it follows that $\Big(d\big((1/n,0),(a,b)\big)\Big)$ is not a null sequence.
Thus in all cases, $\Big(d\big((1/n,0),(a,b)\big)\Big)$ is not a null sequence.

(ii) Consider the sequence $(\mathbf{a}_n)$ where $\mathbf{a}_n = (0,1/n)$ for $n \in \mathbb{N}$.
Then, without loss of generality, for $n < m \in \mathbb{N}$, we have is d-Cauchy, but not d-convergent.
The latter follows from the previous part; thus we need only prove that (a_n) is d-Cauchy.

$$\begin{aligned} d(\mathbf{a}_n, \mathbf{a}_m) &= 3|0-0| + |1/n - 1/m| \\ &= |1/n - 1/m| \\ &\leq 1/n. \end{aligned}$$

Let $\varepsilon > 0$ be given. Then if $N > 1/\varepsilon$, we have, for $m > n > N$

$$d(\mathbf{a}_n, \mathbf{a}_m) \leq 1/n < 1/N < \varepsilon.$$

Hence $(\mathbf{a}_n)$ is a d-Cauchy by Defn 1.3, HB, p135.
Also, by (c)(i) above, $(\mathbf{a}_n)$ is not a d-convergent in X. Thus by Defn 2.1, HB p136, X is not d-complete.

(iii) By the contrapositive of Thm 2.3 (HB p136), since X is not d-complete so X is not d-compact, and, by Thm 5.12 (HB p134), since X is not d-compact, so X is not d-sequentially compact.

Q 7.

Consider the following list of groups.

$\mathbb{Z}_4 \times \mathbb{Z}_5$; $D_5 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_{10}$; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$; Dic_5.

(a)

(i) Dic_5

(ii) $\mathbb{Z}_4 \times \mathbb{Z}_5$

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$

(iv) $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$

(b) We have $|G| = 637 = 17 \times 3^2$.

(i) $3^2 = 9$

(ii) Let m_{17} be the number of Sylow 17-subgroups of order 17. Then from the Sylow Theorems (Thm 1.12, HB p39) we have that the following are all true:

$$\begin{aligned} &m_{17} \geq 1 \\ &m_{17} \equiv 1 \pmod{17} \\ &m_{17} | 9 \end{aligned}$$

So we must have that $m_{17} = 1$. Therefore G has a unique Sylow 17-subgroup or order 17.

Next, let m_3 be the number of Sylow 3-subgroups of order 9. Again, from the Sylow Theorems we have:

$$\begin{aligned} &m_3 \geq 1 \\ &m_3 \equiv 1 \pmod{3} \\ &m_3 | 17 \end{aligned}$$

So we must have that $m_3 = 1$. Therefore G has a unique Sylow 3-subgroup of order 9.

(iii) Let H_1 be the unique subgroup of G of order 9, and let H_2 be the unique subgroup of G of order 17.

By Lemma 1.9, HB p39, each of H_1 and H_2 is normal in G. Further, we have that $|H_1| \times |H_2| = 9 \times 17 = 153 = |G|$. Lastly, we see that $|H_1|$ and $|H_2|$ are coprime. So, $G \cong H_1 \times H_2$, by Thm 2.3, HB p39.

Now, $|H_1| = 3^2$, and so H_1 is abelian by Thm 3.2, HB p36. And since $|H_2|$ is prime, H_2 is cyclic by Prop 3.1, HB p36, and so it is abelian. So we conclude that $H_1 \times H_2$ is abelian, as it is a direct product of abelian groups.

Therefore, since G is isomorphic to an abelian group, it is abelian.

(iv) $\mathbb{Z}_9 \times \mathbb{Z}_{17}$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$

(c) Let G be an infinite group, and let $g \in G$ such that $g \neq e$. Since G is infinite, such a g always exists.

Consider the subgroup generated by g, $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$. Then $\langle g \rangle$ is a non-trivial subgroup of G

If $\langle g \rangle \neq G$, then $\langle g \rangle$ is a non-trivial proper subgroup of G.

If instead $\langle g \rangle = G$, then G is generated by g and thus G is an infinite cyclic group. So $G \cong (\mathbb{Z}, +)$, by Thm 2.1a, HB p31. But $n\mathbb{Z}$ is a proper non-trivial subgroup of $(\mathbb{Z}, +)$ for each $n \in \mathbb{N}, n > 1$, by Cor 2.4, HB p31.
So, G has a non-trivial proper subgroup isomorphic to $2\mathbb{Z}$, i.e $\{g^{2k} : k \in \mathbb{Z}\}$.

So G has a proper non-trivial subgroup in all cases.

Q 8.

$\rho = \frac{1}{2}\left(a + b\sqrt{-3}\right)$

(a) We solve $\rho^2 - \rho + 1 = 0$ using formula for a quadratic equation:

$$\rho = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times 1}}{2} = \frac{1 \pm \sqrt{-3}}{2}.$$

Therefore $\rho = \frac{1}{2}\left(1 + \sqrt{-3}\right)$ satisfies the equation $\rho^2 - \rho + 1 = 0$.
It follows that, in general, $\rho^{n+2} - \rho^{n+1} + \rho^n = 0$ or $\rho^{n+2} = \rho^{n+1} - \rho^n$.
Thus $\rho^k (k \geq 2)$ can be reduced to a combination of 1 and ρ and so every element of $\mathbb{Z}[\rho]$ can be expressed in the form $a + b\rho$.

(b) Let $r = a + b\rho$. For a unit we require $N(r) = 1$, where $N(a + b\rho) = a^2 + ab + b^2$. Now

$$\begin{aligned} a^2 + ab + b^2 = 1 &\Leftrightarrow 4a^2 + 4ab + 4b^2 = 4 \\ &\Leftrightarrow 4a^2 + 4ab + b^2 + 3b^2 = 4 \\ &\Leftrightarrow (2a + b)^2 + 3b^2 = 4. \end{aligned}$$

Therefore $r$is a unit if, and only if, $(2a + b)^2 + 3b^2 = 4$.
If $a = 0$ then $b^2 + 3b^2 = 4$ and so $b = \pm 1$.
If $b = 0$ then $(2a)^2 = 4$ and so $a = \pm 1$.
If $|b| > 1$ then $(2a + b)^2 + 3b^2 > 4$ and there are no solutions in $\mathbb{Z}$.
If $b = 1$ then $(2a + 1)^2 = 1$ and so $a = 0$ or $a = -1$.
If $b = -1$ then $(2a - 1)^2 = 1$ and so $a = 0$ or $a = 1$.
Therefore the solutions of $N(r) = 1$ are:

a	b	r
1	0	1
-1	0	-1
0	1	ρ
0	-1	$-\rho$
1	-1	$1 - \rho$
-1	1	$-1 + \rho$

That is $r = \pm 1, \pm\rho, \pm(1 - \rho)$.

(c) If $N(r) = 3$, where $a^2 + ab + b^2 = 3$. Using the same technique as in part (b), we have $a^2 + ab + b^2 = 3 \Leftrightarrow 4a^2 + 4ab + 4b^2 = 12 \Leftrightarrow (2a + b)^2 + 3b^2 = 12$.
If $|b| > 2$ then $(2a + b)^2 + 3b^2 > 12$ and there are no solutions in $\mathbb{Z}$.
If $b = 0$ then $a^2 = 3$ and there are no solutions in $\mathbb{Z}$.
Therefore the only possible solutions involve $b = \pm 1$ or $b = \pm 2$.
If $b = 1$ then $(2a + 1)^2 = 9$ and so $a = 1$ or $a = -2$.
If $b = -1$ then $(2a - 1)^2 = 9$ and so $a = -1$ or $a = 2$.
If $b = 2$ then $(2a + 2)^2 = 0$ and so $a = 1$.
If $b = -2$ then $(2a - 2)^2 = 0$ and so $a = 1$.

Therefore the solutions of $N(r) = 1$ are:

a	b	r
1	1	$1+\rho$
-2	1	$-2+\rho$
-1	-1	$-1-\rho$
2	-1	$2-\rho$
-1	2	$-1+2\rho$
1	-2	$1-2\rho$

That is $r = \pm(1+\rho), \pm(2-\rho), \pm(1-2\rho)$.

(d) If 3 is reducible to st, then $9 = N(3) = N(s)N(t)$ and, ignoring solutions that involve $N(r) = 1$, we have $N(s) = N(t) = 3$.
Examining solutions to part (c), we try $(1+\rho)(2-\rho) = 2+\rho-\rho^2$.
Recalling that $\rho^2 - \rho + 1 = 0$, we have $\rho - \rho^2 = 1$ and so $(1+\rho)(2-\rho) = 3$.
Thus 3 can be factorized in $\mathbb{Z}[\rho]$.
Since 3 is not a unit, it follows from Defn 4.5 (HB, p51), it is reducible in $\mathbb{Z}[\rho]$.

(e)

(i) From $\rho = \frac{1}{2}\left(1+\sqrt{-3}\right)$ we have $\sqrt{-3} = 2\rho - 1$.
Substituting for $\sqrt{-3}$ in a typical member of R, gives
$a + b\sqrt{-3} = a + b(2\rho - 1) = (a-b) + 2b\rho$.
Also, since $(a-b), 2b \in \mathbb{Z}$ it follows that $a + b\sqrt{-3} \in \mathbb{Z}[\rho]$.
Therefore $R \subseteq \mathbb{Z}[\rho]$.

(ii) 3 is a reducible element in R.
By inspection, $\sqrt{-3} \times -\sqrt{-3} = 3$ and neither $\pm\sqrt{-3}$ are units in R.
In fact $\pm\sqrt{-3} = \pm\,(2\rho - 1)$.
Therefore by Defn 4.5 (HB, p51), 3 is reducible in R.

Q 9.

(a)

(i) Let (f_n) be a sequence in A such that f_n $d_{\max}$ converges to f. We must show that $f \in A$. That is, we must show that

$$|f(x) - f(y)| \leq \left(\sqrt{x} + \sqrt{y}\right)|x - y| \text{ for each } x, y \in [0, 1].$$

Now by the Triangle Inequality (HB, p61), for each $x, y \in [0, 1]$,

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq d_{\max}(f_n,\, f) + \left(\sqrt{x} + \sqrt{y}\right)|x - y| + d_{\max}(f_n,\, f) \end{aligned}$$

since $f_n \in A$.

This inequality is true for arbitrary $n \in \mathbb{N}$ and $d_{\max}(f_n,\, f) \to 0$ as $n \to \infty$, so

$$|f(x) - f(y)| \leq 0 + \left(\sqrt{x} + \sqrt{y}\right)|x - y| + 0 \text{ for each } x, y \in [0, 1],$$

and so, $f \in A$.
Hence A is $d_{\max}$-closed.

(ii) By Appendix 5, HB p89, as B is a closed ball, it is $d_{\max}$-closed. From (i), A is $d_{\max}$-closed. Thus, by Prop 1.17, HB p81, $A \cap B$ is also $d_{\max}$-closed.

(iii) To show that $A \cap B$ is $d_{\max}$-sequentially compact, we use the Arzelà-Ascoli Thm (HB, p131).

1. From (a)(ii) above, $A \cap B$ is $d_{\max}$-closed.
2. From Example 1 of pointwise bounded sets (HB, p130), with g(x) = 2020, B is a pointwise bounded set and so $A \cap B \subseteq B$, is also a pointwise bounded set of functions.
3. We are given that A is an equicontinuous set of functions. Thus, by Example 2 of equicontinuous sets of functions (HB p131), $A \cap B \subseteq A$ is also an equicontinuous set of functions.

The three conditions of the Arzelà-Ascoli Theorem are therefore satisfied and hence $A \cap B$ is $d_{\max}$-sequentially compact.

(b)

(i) Since $\min\{-2, 2\} \leq 0 < \max\{-2, 2\}$, then

$$\begin{aligned} d(-2, 2) &= |\min\{-2, 2\}| + 2\,|\max\{-2, 2\}| \\ &= |-2| + 2\,|2| \\ &= 2 + 2 \times 2 = 6 \end{aligned}$$

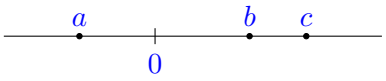
'Write down' so "6" gets the mark.

(ii)

(1) Certainly $d(a,b) \geq 0$ since in all cases the distance function is comprised of addition of positive multiples of moduli.
If $a = b$, then either:
$a = b > 0$ when $\min\{a,b\} = \min\{a,a\} = a > 0$ and so $d(a,b) = 2|a-a| = 0$
or
$a = b \leq 0$ when $\max\{a,b\} = \max\{a,a\} = a \leq 0$ and so $d(a,b) = |a-a| = 0$
Conversely, if $d(a,b) = 0$ then either:
$|\min\{a,b\}| + 2|\max\{a,b\}| = 0 \Rightarrow a = b = 0$
or
$|a-b| = 0 \Rightarrow a = b$ (and $a < 0$)
or
$2|a-b| = 0 \Rightarrow a = b$ (and $a > 0$).
Thus in all cases, $d(a,b) = 0 \Rightarrow a = b$.
Hence (M1) holds.

(2) For any $a, b \in \mathbb{R}$,
$\min\{a,b\} = \min\{b,a\}, \max\{a,b\} = \max\{b,a\}$, and $|a-b| = |b-a|$.
Thus in all cases, $d(a,b) = d(b,a)$. Hence (M2) holds.

(3) Given $a \leq 0$ and $b, c > 0$, without loss of generality, we have the following diagram:



where b, c could be interchanged.
Here $\min\{a,c\} = a$ and $\max\{a,c\} = c$ and so
$d(a,c) = |\min\{a,c\}| + 2|\max\{a,c\}| = |a| + 2c$.
In the same way, $\min\{a,b\} = a$ and $\max\{a,b\} = b$ and so
$d(a,b) = |a| + 2b$. Also $d(b,c) = 2|b-c|$. Thus

$$\begin{aligned} d(a,c) &= |a| + 2c \\ &= |a| + 2b + 2(c-b) \\ &= d(a,b) + 2(c-b) \\ &\leq d(a,b) + 2|b-c| = d(a,b) + d(b,c) \end{aligned}$$

(c) Let (a_n) be an arbitrary d-convergent sequence in X that d-converges to some a in X, such that $a_n \in A \cap B$ for all $n \in \mathbb{N}$. If $a_n \in A \cap B$ then we must have $a_n \in A$ and $a_n \in B$, so it follows that (a_n) is a d-convergent sequence in X such that $a_n \in A$ and $a_n \in B$ for all $n \in \mathbb{N}$. But A is d-closed so, by the definition of d-closed given in the question, the limit a of (a_n) must be in A. By the same argument we have $a \in B$. Now if $a \in A$ and $a \in B$ we must have $a \in A \cap B$, and so (a_n) d-converges to a in $A \cap B$.

We have shown that for any d-convergent sequence in X with all terms in $A \cap B$ the limit of the sequence is also in $A \cap B$. Hence $A \cap B$ is d-closed by the definition, as required.

This is essentially the argument in Prop 1.17 in Chap 16.